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The theory of finite strains of a granular material☆

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Abstract

The flow of a granular material is modelled using the rheological method supplemented by a new element, viz., a rigid contact, which serves to take into account the different resistances of the material to extension and compression. The elastic properties characteristic of the compact medium are taken into account on a phenomenological level. The Cauchy stress tensor and Hencky's logarithmic strain tensor are used to describe the stress-strain state in Euler variables. It is shown that such a choice of tensors ensures thermodynamic correctness of the constitutive relations and leads to a faithful description of the effect of dilatancy for an arbitrary magnitude of the shear.

1. Dilatancy

In the theory of small strains, the constitutive relations of an ideal granular material with rigid particles are formulated in terms of variational inequalities that are equivalent to one another[1]:

$$\sigma : (\tilde{\varepsilon} - \varepsilon) \le 0, \quad \varepsilon, \tilde{\varepsilon} \in C; \quad (\tilde{\sigma} - \sigma) : \varepsilon \le 0, \quad \sigma, \tilde{\sigma} \in K \tag{1.1}$$

In these inequalities, the convex closed cones C and K are used to assign the natural constraints imposed on the tensors of the real strains ε and of the stresses σ that are associated with the fact that the medium is incompressible in the dense state and does not resist extension in the loosened state. Variable quantities are marked with a tilde, and the generally accepted notation and operations of tensor analysis are used.

The constitutive relations take their simplest form in the case of the von Mises–Schleicher circular cone

$$K = \{\sigma | \tau(\sigma) \le \varkappa p(\sigma)\}$$

$$p = -(\sigma_1 + \sigma_2 + \sigma_3)/3, \quad \tau = \left[\sum_{i<j} (\sigma_i - \sigma_j)^2/6\right]^{1/2}, \quad i, j = 1, 2, 3$$

Here p is the hydrostatic pressure, τ is the intensity of the shear stresses, σ_i are the principal stresses, and $\varkappa$ is the internal friction parameter of the medium. It follows from the equivalence condition of variational inequalities (1.1) that the cones C and K are conjugate:

$$C = \{\varepsilon | \sigma : \varepsilon \le 0, \forall \sigma \in K\}, \quad K = \{\sigma | \sigma : \varepsilon \le 0, \forall \varepsilon \in C\}$$

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Therefore, C is also a circular cone of the form

$$C = \{\varepsilon | \varkappa\gamma(\varepsilon) \le \theta(\varepsilon)\}; \quad \theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad \gamma = \left[2\sum_{i<j}(\varepsilon_i - \varepsilon_j)^2/3\right]^{1/2}$$

where θ is the volume strain, and γ is the shear intensity. The limiting equation

$$\varkappa\gamma(\varepsilon) = \theta(\varepsilon)$$

describes the dilatancy, first discovered by Reynolds: shear is accompanied by volume expansion of the medium due to repacking of the particles.

Let us consider some possibilities for generalizing the constitutive relations (1.1) to take into account finite strains and particle rotation in the example of the uniform strain state of combined extension and simple shear. For such a state, the Euler coordinates of the particles are defined in terms of the Lagrangian coordinates using the formulae

$$x_1 = \xi_1, \quad x_2 = \xi_2 + \chi\xi_1, \quad x_3 = \kappa\xi_3 \tag{1.2}$$

Here χ is the tangent of the angle of shear in the x_1, x_2 plane of the Cartesian system of coordinates, and κ is the linear extension coefficient along the x_3 axis. The strain gradient tensor and the inverse tensor have the form

$$\nabla_\xi x = \begin{Vmatrix} 1 & \chi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \kappa \end{Vmatrix}, \quad \nabla_x \xi = \begin{Vmatrix} 1 & -\chi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\kappa \end{Vmatrix}$$

When a medium with rigid particles undergoes finite strains, the permissible strain states are also subject to a certain constraint. We will assume that this constraint can be obtained by replacing the small-strain tensor by the Almansi strain tensor

$$2\varepsilon = \delta - \nabla_x\xi \cdot (\nabla_x\xi)^* = \begin{Vmatrix} -\chi^2 & \chi & 0 \\ \chi & 0 & 0 \\ 0 & 0 & (\kappa^2-1)/\kappa^2 \end{Vmatrix}$$

where δ is the unit tensor and the asterisk denotes transposition. Then, taking into account the expressions for the principal strains

$$4\varepsilon_{1,2} = -\chi^2 \pm \chi\sqrt{4+\chi^2}, \quad 2\varepsilon_3 = (\kappa^2-1)/\kappa^2$$

we can write the dilatancy equation in the form

$$\left[\frac{1}{3}\left(2\frac{\kappa^2-1}{\kappa^2} + \chi^2\right)^2 + \chi^2(4+\chi^2)\right]^{1/2} = \frac{1}{\varkappa}\left(\frac{\kappa^2-1}{\kappa^2} - \chi^2\right) \tag{1.3}$$

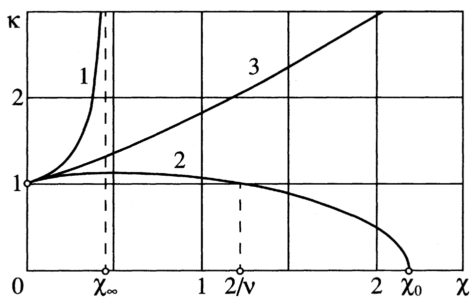

Fig. 1.

In the χ, κ plane (Fig. 1) this equation corresponds to monotonic curve 1 with the vertical asymptote $\chi = \chi_\infty$. If 103d, the value of $\chi_\infty \leq 1$ is determined as the solution of the equation that follows from (1.3) when $\kappa \to \infty$. The tangent to the curve at the point $\chi = 0$ is assigned by the dilatancy equation of the theory of small strains

$$\chi = \nu(\kappa - 1), \quad \nu = \left[\frac{1}{\varkappa^2} - \frac{4}{3}\right]^{1/2}$$

If $\varkappa \geq \sqrt{3}/2$, the dilatancy curve degenerates to a point. “Jamming” of the particles is observed: in such a medium, displacement is impossible without preliminary volume expansion.

An obvious shortcoming of a description of dilatancy based on the Almansi finite strain tensor is that the angle of shear in a dilatant medium cannot exceed the limit value arctan $\chi_\infty \leq \pi/4$, and when this value is attained, the volume of the medium tends to infinity. We will assume that the constraint can be formulated in terms of the Cauchy–Green strain tensor

$$2\epsilon = \nabla_\xi x \cdot (\nabla_\xi x)^* - \delta = \begin{Vmatrix} \chi^2 & \chi & 0 \\ \chi & 0 & 0 \\ 0 & 0 & \kappa^2 - 1 \end{Vmatrix}$$

for which the principal strains have the form

$$4\epsilon_{1,2} = \chi^2 \pm \chi\sqrt{4 + \chi^2}, \quad 2\epsilon_3 = \kappa^2 - 1$$

Then the dilatancy equation takes the following form

$$[(2\kappa^2 - 2 - \chi^2)^2/3 + \chi^2(4 + \chi^2)]^{1/2} = (\kappa^2 - 1 + \chi^2)/\varkappa \tag{1.4}$$

In Fig. 1 it corresponds to curve 2. According to Eq. (1.4), the expansion of the medium due to shear is replaced by compression when $\chi = 2/\nu$, and the volume tends to zero as $\chi \to \chi_0 \geq 1$. The limit value χ_0 can be found as the root of Eq. (1.4) with $\kappa = 0$, but the range of angles of shear in which dilatancy is described satisfactorily is bounded by a value significantly smaller than arctan χ_0.

We will use Hencky's logarithmic strain tensor $h = \ln\sqrt{l}$, where $l = (\nabla_\xi x)^* \cdot \nabla_\xi x$ is the left Cauchy–Green tensor, to formulate of the constraints under consideration. Some unique properties of the logarithmic tensor were established in Refs. 2,3. In particular, it was proved that it is a unit Euler tensor, which is related to the Cauchy strain-rate tensor $e = (\nabla_x v + (\nabla_x v)^*)/2$ by a co-rotation differentiation operation:

$$e = Q^* \cdot \frac{d}{dt}(Q \cdot h \cdot Q^*) \cdot Q = \dot{h} + h \cdot \Omega - \Omega \cdot h, \quad Q \cdot Q^* = \delta, \quad \Omega = \dot{Q}^* \cdot Q$$

(the dot over a letter denotes the total derivative with respect to time). The skew-symmetric tensor Ω, which is the spin of this derivative, was also obtained in explicit form. The energy compatibility of the logarithmic tensor and the Cauchy stress tensor is thereby established. The general form of the constitutive relations of an isotropic hyperelastic medium was found:

$$\sigma = \rho\partial\Phi/\partial h \tag{1.5}$$

where ρ is the density and Φ is the elastic potential. It was shown in Ref. 4 that the use of the quadratic potential of the linear theory of elasticity, written in terms of the invariants of the logarithmic tensor, enables the behaviour of an elastic material in the range of moderate strains to be described with satisfactory accuracy. In addition, the additive decomposition of Hencky's tensor into the deviator and the spherical part is known to correspond to a representation of the strain process in the form of the superposition of the deformation and bulk strain of the medium. The continuity equation characterizing the bulk strain reduces to the equation

$$\theta(h) = \ln\frac{\rho_0}{\rho} \tag{1.6}$$

where ρ_0 is the initial density of the medium

In a state of combined extension and simple shear, we have

$$l = \begin{Vmatrix} 1 & \chi & 0 \\ \chi & 1+\chi^2 & 0 \\ 0 & 0 & \kappa^2 \end{Vmatrix}$$

The principal stresses can be found in terms of the eigenvalues of this tensor:

$$h_i = \ln\sqrt{l_i}, \quad 2l_{1,2} = 2+\chi^2 \pm \chi\sqrt{4+\chi^2}, \quad l_3 = \kappa^2$$

Using the expressions

$$\theta(h) = \ln\kappa, \quad \gamma(h) = \left[\ln^2 l_1 + \frac{4}{3}\ln^2\kappa\right]^{1/2}$$

we can write the dilatancy equation in the form $l_1 = \kappa^{\nu}$ or, after solving it for χ, in the form

$$\chi = \kappa^{\nu/2} - \kappa^{-\nu/2} \tag{1.7}$$

In Fig. 1, curve 3 corresponds to this equation.

According to (1.7), for a constant value of ν, the volume of the medium expands without limit as the angle of shear increases. In fact, the internal friction parameter depends on the density and tends to zero as a certain critical value ρ_* is reached. The dilatancy curve then approaches the horizontal asymptote $\kappa = \kappa_*$. The value of κ_* for real granular materials is determined by the size and shape of the particles. By measuring it on an experimental system, it can easily be found that $\rho_* = \rho_0/\kappa_*$.

Fine-grained materials, in which the dilatancy-induced density change is small, can be modelled using the power-law dependence

$$\varkappa = \begin{cases} \varkappa_0\left(\dfrac{1-\rho_*/\rho}{1-\rho_*/\rho_0}\right)^n, \text{ if } \rho \ge \rho_* \\ 0, \text{ if } \rho < \rho_* \end{cases} \tag{1.8}$$

A series of dilatancy curves (1.7) with the dependence (1.8) for 105d, $\kappa_* = 1.25$ and various values of n is presented in Fig. 2. Satisfactory approximation of the experimental curves can be achieved by adjusting the values of the coefficients $\varkappa_0$ and n.

It should be noted that consideration of the dependence of $\varkappa$ on the density does not eliminate the aforementioned errors in models that employ the Almansi and Cauchy–Green finite strain tensors, which lead to fairly rigid constraints on the magnitude of the shear.

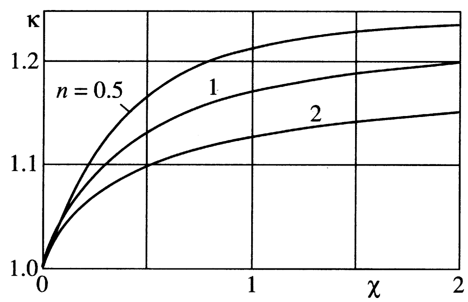

Fig. 2.

2. The mathematical model

We will construct the constitutive relations of the isothermal strain of a granular material in accordance with the rheological scheme shown in Fig. 3a. Under compressive stresses, the rigid contact appearing in the scheme is not deformed, and the medium behaves as an absolutely rigid body. Under extension, the stress in the contact is equal to zero; therefore, the behaviour of the medium is described by the model of a viscous fluid. Since the stress field in an absolutely rigid body is not uniquely specified in the general case, this model is ill-posed. One of the regularization techniques is to add an elastic element that takes into account the compliance of the particles (see Fig. 3b). According to the regularized scheme, the state of the medium under compression is characterized by the Kelvin-Voight viscoelasticity model. A viscous element also plays a regularizing role in the schemes under consideration. Without it the model becomes ill-posed: the tensile strain field is not uniquely defined.

The mechanism underlying the appearance of viscous stresses in a loosened granular material moving in a fluid flow was investigated experimentally in Ref. 6, which provided the basis for the theory of rapid motions.[7–9] It was found that the significant dispersive pressure in such a medium, which is caused by the contact interaction of the rigid particles, and the shear stress, which is proportional to this pressure, are present even in the case of a weakly viscous fluid. Models of the quasi-static stress-strain state of a close-packed granular material were proposed in Refs. 10–12. Models based on the rheological schemes in Fig. 3 can be used to describe mixed flows, in which regions of rapid motions are combined with dead zones of quasi-static strain. Taking into account the disperse nature of viscous stresses, we can also conjecture that these models are capable of describing the flow of moderately moist and dry materials with satisfactory accuracy. However, the question of determining the phenomenological coefficient of viscosity as a function of the degree of loosening of the material and the viscous properties of the fluid that can be present in the pore space is fairly complex and will not be considered here.

We will first postulate that the coefficient $\varkappa$ is constant. Taking into account the energy compatibility of Hencky's strain tensor and the Cauchy stress tensor,[2] we will present the constitutive relations for the scheme in Fig. 3a in the form

$$\sigma = \sigma^{v} + \sigma^{c}, \quad \sigma^{v} = (m - 2\eta/3)\theta(e)\delta + 2\eta e, \quad \sigma^{c} : (\tilde{h} - h) \le 0, \quad h, \tilde{h} \in C \tag{2.1}$$

Here σ^{v} is the viscous stress tensor, which is defined in terms of the strain-rate tensor by Stokes law, and m and η are the bulk and shear coefficients of viscosity, which depend on the density of the medium in the general case. In the case of small strains, relations (1.1), which correspond to a rheological scheme consisting of one element, viz., a rigid contact, can be obtained from (2.1) when m, $\eta \to 0$.

According to the Kuhn–Tucker theorem, the variational inequality (2.1) reduces to the problem of maximizing the Lagrangian with respect to $\tilde{h}$:

$$\sigma^{c} : \tilde{h} - \lambda(\varkappa\gamma(\tilde{h}) - \theta(\tilde{h}))$$

The non-negative Lagrange multiplier λ appearing therein is equal to zero if $\varkappa\gamma(h) < \theta(h)$. In this case, a regime of viscous flow of the medium with $\sigma = \sigma^{v}$ occurs. In the rigid zones, where $h = 0$, the condition for a maximum

$$\sigma^{c} : \tilde{h} \le 0, \quad \forall \tilde{h} \in C$$

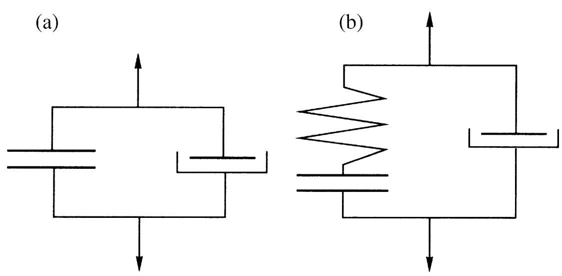

Fig. 3.

means that $\sigma^c \in K$. Here the stress tensor and the multiplier λ remain non-negative. Of course, if $\varkappa\gamma(h) = \theta(h)$ and $h \neq 0$, then from the maximization condition of the Lagrangian in differential form we obtain

$$\frac{\sigma^c}{\lambda} = -\left(1 + \frac{2\varkappa^2}{3}\right)\delta + \frac{2\varkappa}{\gamma(h)}h \tag{2.2}$$

Hence it follows that $\tau(\sigma^c) = \varkappa p(\sigma^c)$, i.e., the medium is in the limit state, and that the deviators of the tensors σ^c and h are proportional.

The variational inequality (2.1) can be reduced to the equivalent potential form

$$\sigma^c \in \rho\partial\delta_C(h), \quad \delta_C(h) = \begin{cases} 0, & \text{if} \quad h \in C \\ +\infty, & \text{if} \quad h \notin C \end{cases}$$

Here $\delta_c(h)$ is the indicator function of the cone C, and the symbol ∂ is used to denote the subdifferential

$$\partial\Phi(h) = \{s | \Phi(\tilde{h}) - \Phi(h) \geq s : (\tilde{h} - h), \forall\tilde{h}\}$$

which describes the set of angular coefficients of the linear functions whose graphs pass through the point $(h, \Phi(h))$ and lie below the graph of the function Φ.

The potential form indicates out that in the case of quasi-static strain, in which the viscous stresses are small, the constitutive relations (2.1) describe a thermodynamically reversible process and apply to the non-linear theory of elasticity. To some extent, reversibility contradicts the models of internal friction between the particles. Therefore, the permissible shear stresses in a granular material should probably be associated with surface tension, electromagnetic forces, etc., rather than with friction. Unlike the traditional non-linear elastic model (1.5), in our case the potential is not a differentiable function. The use of a subdifferential, which provides a natural generalization of a derivative, is thus related to the degeneration of the potential. This situation is characteristic of models of the theory of plasticity, in which the associated law of plastic flow is formulated in terms of subdifferentials.[13]

The dual strain potential of a granular material is calculated using the Young transform

$$\Psi(s) = \sup_{\tilde{h}}\{s : \tilde{h} - \Phi(\tilde{h})\}$$

Since the Young transform of the indicator function of a convex cone is the indicator function of the conjugate cone, inequality (2.1) can also be reduced to the form $h \in \partial\delta_K(\sigma^c/\rho)$.

Let k and μ be the bulk modulus and the shear modulus of a compact medium. Taking into account the additivity of the expansion of the strain tensor at the joint between the rigid contact and the elastic element of the scheme in Fig. 3b, we obtain the constitutive relations of a granular material with elastic properties

$$h \in \partial\Psi(\sigma^c/\rho); \quad \Psi(s) = \rho_0|s|^2/2 + \delta_K(s), \quad |s| = \sqrt{p^2(s)/k + \tau^2(s)/\mu} \tag{2.3}$$

Here $|s|$ is the energy norm of the tensor. Letting k and μ tend to infinity, we can obtain the constitutive relations (2.1) for a material with rigid particles from the relations (2.3).

Physically, the linear Hooke's law

$$\rho_0 s = (k - 2\mu/3)\theta(h)\delta + 2\mu h$$

establishes a one-to-one correspondence between the spaces of the tensors s and h. The symmetric bilinear form

$$(\tilde{s}, s) = \tilde{s} : h/\rho_0 = s : \tilde{h}/\rho_0$$

assigns a scalar product matched to the norm introduced in the space of s.

By virtue of the involute nature of the Young transform, the stress potential is found using the formula

$$\Phi(h) = \sup_{\tilde{s}}\{\tilde{s} : h - \Psi(\tilde{s})\}$$

Thus,

$$\Phi(h)/\rho_0 = \sup_{\tilde{s}\in K}\{(\tilde{s}, s) - |\tilde{s}|^2/2\} = \sup_{\tilde{s}\in K}\{|s|^2 - |s-\tilde{s}|^2\}/2 =$$
$$= |s|^2/2 - \inf_{\tilde{s}\in K}|s-\tilde{s}|^2/2 = |s|^2/2 - \left|s - s^\pi\right|^2/2$$

where the superscript π denotes the projection of the tensor onto the cone K. Hence, since the scalar product $(s^\pi, s - s^\pi)$ is equal to zero in the case of projection onto a cone (see, for example, Ref. 14), we obtain

$$\Phi(h) = \rho_0\left|s^\pi\right|^2/2$$

Using the formula for the derivative of the expression $|s-s^\pi|^2$ (Ref. 14, p. 222), we can show that the potential is a continuously differentiable function and that

$$\sigma^c = \rho\partial\Phi/\partial h = \rho s^\pi \tag{2.4}$$

The projection onto a Mises–Schleicher cone is determined in the following manner.[15] If $\tau(s) \le \varkappa p(s)$ for $s \in K$, the equality $s^\pi = s$ holds. In this case, by (2.4), the medium deforms according to the Kelvin-Voight viscoelasticity law. If $\mu p(s) + \varkappa k\tau(s) \le 0$ for $h \in C$, the apex of the cone $s^\pi = 0$ serves as the projection required, and viscous flow occurs. When

$$\tau(s) > \varkappa p(s), \quad \mu p(s) + \varkappa k\tau(s) > 0$$

the projection belongs to a conical surface and can be calculated using the formulae describing the limiting strain regime:

$$p(s^\pi) = \frac{\mu p(s) + \varkappa k\tau(s)}{\mu + \varkappa^2 k}, \quad \frac{s^\pi}{p(s^\pi)} = -\left(1 - \frac{\varkappa p(s)}{\tau(s)}\right)\delta + \frac{\varkappa}{\tau(s)}s$$

Since by Hooke's law

$$p(s) = -k\theta(h)/\rho_0, \quad \tau(s) = \mu\gamma(h)/\rho_0$$

we have

$$2\rho_0\Phi = \begin{cases} 0, & \text{if } \varkappa\gamma(h) \le \theta(h) \\ k\theta^2(h) + \mu\gamma^2(h), & \text{if } \mu\gamma(h) + \varkappa k\theta(h) \le 0 \\ \dfrac{\mu k}{\mu + \varkappa^2 k}(\varkappa\gamma(h) - \theta(h))^2, & \text{if } \varkappa\gamma(h) > \theta(h) \text{ and } \mu\gamma(h) + \varkappa k\theta(h) > 0 \end{cases}$$

As k, $\mu \to \infty$, the potential tends to the indicator function $\delta_C(h)$; therefore, the tensor Eq. (2.4) transforms into the variational inequality (2.1).

Taking into account the dependence of the coefficient $\varkappa$ on the density, we rewrite the relation (2.4) using the formula $\partial\rho/\partial h = -\rho\delta$, which follows from continuity Eq. (1.6), in the more general form

$$\sigma^c = \rho\left(\left.\frac{\partial\Phi}{\partial h}\right|_\varkappa + \left.\frac{\partial\Phi}{\partial\varkappa}\right|_h\frac{\partial\varkappa}{\partial h}\right) = \rho s^\pi - q\delta$$
$$q = \frac{\mu k}{(\mu + \varkappa^2 k)^2}(\mu\gamma(h) + \varkappa k\theta(h))_+(\varkappa\gamma(h) - \theta(h))_+\rho^2\frac{d\varkappa}{d\rho} \tag{2.5}$$

Here q is the corrective pressure due to the dependence of $\varkappa$ on ρ, and the subscript plus denotes the positive part of the expression: $z_+ = (z + |z|)/2$. Note that q is non-zero only in the limit, where, according to (2.5), the stress tensor $\sigma^c + q\delta$ applies to the conical surface K:

$$\tau(\sigma^c)/\varkappa = p(\sigma^c) - q \tag{2.6}$$

It can be shown that the constitutive relations that take into account the elastic and viscous properties of a granular material are thermodynamically correct: as a consequence of the isotropy of the medium, the following heat flow equation holds for them

$$\sigma : e = (Q\cdot(\sigma-\sigma^{\upsilon})\cdot Q^*) : \frac{d}{dt}(Q\cdot h\cdot Q^*) + \sigma^{\upsilon} : e =$$
$$= \rho\frac{\partial\Phi}{\partial(Q\cdot h\cdot Q^*)} : \frac{d}{dt}(Q\cdot h\cdot Q^*) + \sigma^{\upsilon} : e = \rho\dot{\Phi}(h) + m\theta^2(e) + \eta\gamma^2(e)$$

in which the term $\rho\dot{\Phi}(h)$ is the rate of change of the reversible strain potential energy, and the sum of the last two terms is the power of the dissipative forces. Since $s^{\pi}\in K$, the inequality $s^{\pi} : \tilde{h} \le 0$ holds for any $\tilde{h}\in C$, and, in addition, according to the equation $(s^{\pi}, s-s^{\pi})=0$,

$$s^{\pi} : h = \rho_0(s^{\pi}, s) = \rho_0|s^{\pi}|^2 \ge 0$$

Therefore, by virtue of (2.5),

$$\sigma^c : (\tilde{h}-h) + q(\theta(\tilde{h})-\theta(h)) \le 0, \quad \forall\tilde{h}\in C \tag{2.7}$$

Hence, letting k and μ tend to infinity, we obtain an extension of variational inequality (2.1) to the case of the variable $\varkappa$. This extension is identical with the inequality (2.7) apart from the replacement of the corrective pressure q by q_{∞}.

In the limit state of the medium, the tensor equality (2.5) reduces to the condition of proportionality of the deviators of the tensors σ^c and h and the equation for the pressure

$$p(\sigma^c) - q = \frac{\rho}{\rho_0}\frac{\mu k}{\mu+\varkappa^2 k}(\varkappa\gamma(h)-\theta(h))$$

The expression for q can be reduced to the following form

$$q = \frac{p(\sigma^c)-q}{\mu+\varkappa^2 k}(\mu\gamma(h)+\varkappa k\theta(h))\rho\frac{d\varkappa}{d\rho}$$

Hence, taking into account that $\varkappa\gamma(h)\to\theta(h)$ as $k, \mu\to\infty$, we obtain

$$q_{\infty} = \frac{f-1}{f}p(\sigma^c), \quad f = 1+\frac{\rho}{\varkappa}\frac{d\varkappa}{d\rho}\ln\frac{\rho_0}{\rho}$$

and from the limit state condition (2.6) it follows that

$$\tau(\sigma^c) = \varkappa p(\sigma^c)/f \tag{2.8}$$

Application of the Kuhn–Tucker theorem to the variational inequality (2.7) leads to the equation obtained by replacing σ^c by $\sigma^c + q_{\infty}\delta$ in equality (2.2).

A closed, thermodynamically correct model of the dynamics of a granular material is provided by constitutive relations (2.5) supplemented by the tensor equality (2.1) for the viscous stresses, the equations of motion and the kinematic equations that relate the strain tensor to the velocity vector (g is the volume force vector):

$$\rho\dot{\upsilon} = \nabla\cdot\sigma + \rho g, \quad h = \ln\sqrt{l}, \quad l = (\nabla_{\xi}x)^*\cdot\nabla_{\xi}x, \quad \dot{x} = \upsilon$$

To calculate the logarithmic tensor for an assigned motion of a granular material, we must solve the characteristic equation

$$l^3 - al^2 + bl - c = 0$$

$$a = \theta(l), \quad b = \frac{\theta^2(l)-\theta(l^2)}{2}, \quad c = \frac{\theta^3(l)}{6} - \frac{\theta(l)\theta(l^2)}{2} + \frac{\theta(l^3)}{3}$$

for the left Cauchy–Green tensor.

According to Cardano's formulae,

$$l_i = \frac{1}{3}\left(a + 2\sqrt{a^2 - 3b}\cos\frac{\beta - 2\pi i}{3}\right), \quad \beta = \arccos\frac{2a^3 - 9ab + 27c}{2(a^2 - 3b)^{3/2}}$$

According to the Lagrange–Sylvester formula for isotropic tensor functions,

$$h = A\delta + Bl + Cl^2$$

where A, B and C are scalar coefficients that depend on invariants of the tensor l. Changing to the principal axes, we can show that these coefficients are the coefficients of a second-order interpolation polynomial that takes the values $h_i = \ln\sqrt{l_i}$ at the nodes l_i. If any of the l_i coincide, an interpolation problem with multiple nodes appears. Its solution is the Hermite polynomial

$$h = h_1\delta + h_{1;\,2}(l - l_1\delta) + h_{1;\,2;\,3}(l - l_1\delta)(l - l_2\delta)$$
$$h_{i;\,j} = \frac{h_j - h_i}{l_j - l_i}, \quad h_{1;\,2;\,3} = \frac{h_{2;\,3} - h_{1;\,2}}{l_3 - l_1} \tag{2.9}$$

Here $h_{i;j}$ and $h_{1;\,2;\,3}$ are the separated first- and second-order differences. When the interpolation nodes coincide, these differences are found in terms of the first and second derivatives with respect to $h_{i;\,j}$ using the formulae

$$h_{i;\,i} = 1/(2l_i), \quad h_{i;\,i;\,i} = -1/(4l_i^2)$$

An alternative method for determining the logarithmic tensor leads to direct calculation of the tensor function by making the transition to the principal axes l. However, this method may be less convenient in practice, since its realization requires the construction of the transition matrix in an explicit form.

3. Shear stresses

The proposed model is universal and sufficiently simple for computer calculations. In addition, because it neglects the elasticity of the particles, it can be used to investigate some exact solutions. As an example, we will determine the stress state of an ideal granular material with rigid particles under uniform shear after preliminary compression in the axial direction by a constant pressure p_0. Such a state appears, for example, in the experimental system described in Ref. 5 or in the rotational motion of a material between two coaxial cylinders of large radius. The strain is described by kinematic Eq. (1.2). By virtue of (1.7), we have

$$l_1 = \kappa^{\nu}, \quad l_2 = \kappa^{-\nu}, \quad l_3 = \kappa^2, \quad h_1 = -h_2 = \frac{\nu}{2}\ln\kappa, \quad h_{1;\,2} = \frac{\nu}{\kappa^{\nu} - \kappa^{-\nu}}\ln\kappa$$

The tensors $l - l_1\delta$ and $l - l_2\delta$ in (2.9) are calculated using the formula

$$l - l_{1,\,2}\delta = \begin{Vmatrix} 1 - \kappa^{\pm\nu} & \kappa^{\nu/2} - \kappa^{-\nu/2} & 0 \\ \kappa^{\nu/2} - \kappa^{-\nu/2} & \kappa^{\mp\nu} - 1 & 0 \\ 0 & 0 & \kappa^2 - \kappa^{\pm\nu} \end{Vmatrix}$$

Their product equals

$$(l - l_1\delta)\cdot(l - l_2\delta) = \operatorname{diag}\{0, 0, (\kappa^2 - \kappa^{\nu})(\kappa^2 - \kappa^{-\nu})\}$$

The components of Hencky's logarithmic tensor in a Cartesian system of coordinates is found from expansion (2.9) and have the form

$$h_{11} = -h_{22} = -\frac{\nu}{2}\frac{\kappa^{\nu/2} - \kappa^{-\nu/2}}{\kappa^{\nu/2} + \kappa^{-\nu/2}}\ln\kappa, \quad h_{12} = \frac{\nu\ln\kappa}{\kappa^{\nu/2} + \kappa^{-\nu/2}}, \quad h_{33} = \ln\kappa \tag{3.1}$$

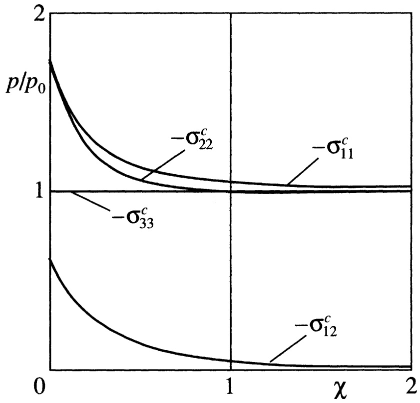

Fig. 4.

If the strain program is specified by the equation $\chi=\chi(t)$, the projections of the velocity vector and the non-zero components of the strain-rate tensor can be calculated by differentiating (1.2):

$$v_1 = 0, \quad v_2 = \dot{\chi}x_1, \quad v_3 = \dot{\kappa}x_3/\kappa, \quad 2e_{12} = \dot{\chi}, \quad e_{33} = \dot{\kappa}/\kappa$$

The tensor of the viscous stresses in the medium is determined from the formulae

$$\sigma_{11}^v = \sigma_{22}^v = (m-2\eta/3)\dot{\kappa}/\kappa, \quad \sigma_{12}^v = \eta\dot{\chi}, \quad \sigma_{33}^v = (m+4\eta/3)\dot{\kappa}/\kappa \tag{3.2}$$

Using (3.1), as well as the condition for proportionality of the deviators, that follows from the equality (2.2), the limiting surface Eq. (2.8) and the condition $\sigma_{33}=-p_0$, we finally obtain

$$\frac{\sigma_{kk}^c}{p(\sigma^c)} = \left((-1)^k\nu\frac{\kappa^{\nu/2}-\kappa^{-\nu/2}}{\kappa^{\nu/2}+\kappa^{-\nu/2}}-\frac{2}{3}\right)\frac{\varkappa^2}{f}-1, \quad k = 1,2$$
$$\frac{\sigma_{12}^c}{p(\sigma^c)} = \frac{2\nu\varkappa^2}{(\kappa^{\nu/2}+\kappa^{-\nu/2})f}, \quad p(\sigma^c) = \frac{p_0+\sigma_{33}^v}{1-4\varkappa^2/(3f)} \tag{3.3}$$

According to this result, as the angle increases, the deviator components of the tensor σ^c relax, and a state of viscous shear occurs in the limit. At the same time, the normal stresses in the medium tend to $-p_0$. Graphs of the quasi-static stresses as a function of χ are shown in Fig. 4. They were constructed using expressions (1.8) with $n=0.5$ and the corresponding expressions for f:

$$f = 1+\frac{n\rho_*}{\rho-\rho_*}\ln\frac{\rho_0}{\rho}$$

According to Fig. 4, the shear stress σ_{12}^c, which is equal to zero until the onset of strain, undergoes a discontinuous change when $\chi=0$. This is because the elastic properties of the medium are not taken into account in the model under consideration. Actually, the dilatancy is preceded by an elastic stage of strain without a change in volume, in which there is a monotonic increase in the shear stress. The elastic stage can be described within the small strain theory. This is confirmed by the agreement between the initial stresses calculated from formulae (3.3) as $\kappa\to 1$ and the limit shear stresses of an elastic granular material under small strains[15]:

$$\sigma_{11} = \sigma_{22} = -p_0(1+2/\nu_0^2), \quad \sigma_{12} = p_0/\nu_0 \tag{3.4}$$

Note that solution (3.3) can be used to analyse the stress state of a granular material under uniform shear taking its own weight into account. In this case, taking into account that $dx_3=\kappa d\xi_3$, that $\rho\kappa=\rho$ and that κ does not depend on x_3, from the equilibrium equation $\partial p_0/\partial x_3=-\rho g$ we find

$$p_0 = \rho g(a_0\kappa - x_3)$$

where a_0 is the initial thickness of the layer. Thus, the viscous stresses (3.2) are constant, and the quasi-static stresses are linear with respect to depth.

4. Couette flow

Dead zones of quasi-static strain appear, for example, in the rotational motion of a close-packed granular material in the space between two extended coaxial cylinders. We will assume that the inner cylinder of radius r_0 rotates about its own axis with a constant angular velocity ω_0 and that the radius of the outer fixed cylinder is infinitely large. If the friction of the medium against the wall is sufficiently large, a transient shear motion, which is non-uniform over the radius and is accompanied by dilatancy, occurs in a certain region near the inner cylinder in the initial stage. Volume expansion leads to displacement of the particles in the axial direction and consequently to a rise in pressure in this region. A steady regime, in which the portion of the material that is far from the rotating cylinder remains fixed and the density of the medium reaches the critical value ρ_* in the moving region, is established with time.

A detailed description of the strain process within the proposed mathematical model is only possible using numerical methods. Neglecting the weight of the granular material and the influence of the inertial forces, we construct a very simple steady solution of the slow-motion problem, which does not depend on the axial coordinate z. Let r_1 be the radius of the unknown boundary that separates the viscous flow region from the dead zone, let p_0 be magnitude of the initial pressure in the axial direction, and let M_0 be the torque applied to a unit length of the inner cylinder.

Flow with $r_1 > r_0$ only occurs if M_0 is above the limit value $M_0^* = 2\pi r_0^2 p_0/\nu_0$, at which the stress state (3.4) occurs on the inner cylinder. In a strict sense, this state, which corresponds to the onset of dilatancy, must be determined for the infinitesimal layer of the granular material adjacent to the cylinder of radius r_0. However, such a problem again leads to formulae (3.4), apart from the replacement of the Cartesian system of coordinates by the cylindrical system r, φ, z.

In the case of steady motion, the velocity vector of the particles and the shear rate are determined in terms of the local angular velocity $\omega(r)$ using the formulae

$$v_r = v_z = 0, \quad v_\varphi = \omega r, \quad 2e_{r\varphi} = \partial v_\varphi/\partial r - v_\varphi/r = r\partial\omega/\partial r \tag{4.1}$$

The remaining components of the strain-rate tensor are equal to zero. The non-zero components of the stress tensor satisfy the equilibrium equations

$$\partial\sigma_r/\partial r + (\sigma_r - \sigma_\varphi)/r = 0, \quad \partial\sigma_{r\varphi}/\partial r + 2\sigma_{r\varphi}/r = 0, \quad \partial\sigma_z/\partial z = 0 \tag{4.2}$$

The parameter $\varkappa$ takes a zero value in the flow region; therefore, according to condition (2.8), the intensity of the shear stresses for the tensor σ^c is also equal to zero, i.e., this tensor is spherical. By virtue of (2.1) and (4.1), we have

$$\sigma_r = \sigma_\varphi = \sigma_z = -p_1, \quad \sigma_{r\varphi} = \eta r\partial\omega/\partial r$$

The first and third equations in (4.2) are satisfied automatically if $p_1 = \text{const}$. After integrating the second equation, taking the boundary conditions for ω at $r = r_0$ and $r = r_1$ into account, we find

$$\sigma_{r\varphi} = -2\eta A\frac{r_1^2}{r^2}, \quad \omega = A\left(\frac{r_1^2}{r^2} - 1\right), \quad A = \frac{\omega_0 r_0^2}{r_1^2 - r_0^2} \tag{4.3}$$

Assuming that the medium is in the limit state (3.4) with $\sigma_r = \sigma_{11}$, $\sigma_\varphi = \sigma_{22}$ and $\sigma_{r\varphi} = \sigma_{12}$ at the interface, from the conditions of continuity of σ_r and $\sigma_{r\varphi}$ we can find the position of the unknown boundary and the magnitude of the pressure:

$$r_1 = r_0\sqrt{1 + 2\eta\omega_0\nu_0/p_0}, \quad p_1 = p_0(1 + 2/\nu_0^2)$$

In the dead zone, the stresses are statically indeterminate. To calculate them, we must change to a model that takes into account the elastic properties of the medium and then let the moduli of elasticity k and μ tend to infinity. Since the strains h_r and h_φ are identical and equal to zero in this zone, we have $\sigma_r = \sigma_\varphi$ by Hooke's law. In accordance with Eq. (4.2), the stress σ_r is constant. Thus, in the dead zone $\sigma_r = \sigma_\varphi = -p_1$, $\sigma_z = -p_0$, and $\sigma_{r\varphi}$ is given by (4.3). It is proved that the stress tensor obtained is permissible, i.e., it applies to the cone K for any $r > r_1$.

Since $M_0 = -2\pi r_0^2\sigma_{r\varphi}(r_0)$, we have the formulae

$$\frac{1}{\nu_0} = \frac{M_0}{2\pi r_1^2 p_0}, \quad \eta = \frac{M_0}{4\pi\omega_0}\left(\frac{1}{r_0^2} - \frac{1}{r_1^2}\right) \tag{4.4}$$

which can be used for experimentally determining the internal friction parameter and the coefficient of viscosity of the granular material from measurements of the angular velocity, the torque and the width of the viscous flow region. One specific feature of such an experiment is that the pressure along the z axis in the flow region depends on ν_0.

It is noteworthy that if the viscosity is small, the width of the flow region is proportional to η:

$$\frac{r_1 - r_0}{r_0} \approx \eta\omega_0\frac{\nu_0}{p_0}$$

As the coefficient of viscosity tends to zero, the flow degenerates, and the rotating cylinder slips, as in the case of an absolutely smooth surface. As the pressure decreases, the dead zone narrows and vanishes completely when $p_0 \to 0$, while an increase in pressure has the same effect on the position of the boundary as does a decrease in the viscosity of the medium.

A solution that describes the steady flow of a granular material between coaxial cylinders taking the rotational inertia into account, i.e., a right-hand side of the form $-\rho\omega^2 r$ in the first equation in (4.2), is constructed in a similar manner. The only difference is in the expression for the pressure in the region $r_0 < r < r_1$, which is found from the equation

$$p = p_1 + \rho_*\int_{r_1}^{r}\omega^2 r dr = p_1 - \frac{\rho_* A^2}{2}\left(\frac{r_1^4}{r^2} + 4r_1^2\ln\frac{r}{r_1} - r^2\right) \tag{4.5}$$

When $p_0 \to 0$ $(r_1 \to \infty)$, we can use relations (4.3) and (4.5) and the auxiliary limits

$$A \to 0, \quad Ar_1^2 \to \omega_0 r_0^2, \quad A^2 r_1^2 \ln r_1 \to 0$$

to obtain a solution with an infinite flow region

$$\sigma_{r\varphi} = -2\eta\omega_0 r_0^2/r^2, \quad \omega = \omega_0 r_0^2/r^2, \quad p = \rho_*\omega_0^2 r_0^4/(2r^2) \tag{4.6}$$

Taking the limit as $\eta \to 0$ in (4.6) reveals that regardless of the magnitude of the angular velocity, non-viscous steady flow occurs for a torque $M_0 = 4\pi r_0^2\eta\omega_0$, which tends to zero. However, the kinematic characteristics of motion remain unchanged when this transition is made.

5. Pressure on an inclined plane

Let us investigate the slow unsteady motion of a layer of a granular material under the influence of gravity on a rough inclined plane with a slope α to the horizontal plane in a plane formulation. Under the condition of adhesion, the kinematics of the motion are given by the equations

$$dx_1 = d\xi_1 + \kappa\chi d\xi_2, \quad dx_2 = \kappa d\xi_2, \quad dx_3 = d\xi_3 \tag{5.1}$$

in which the quantities $\kappa(t, x_2)$ and $\chi(t, x_2)$ have their former meanings (see Fig. 5). The strain-gradient tensor and the left Cauchy–Green tensor take the following forms

$$\nabla_\xi x = \begin{Vmatrix} 1 & 0 & 0 \\ \kappa\chi & \kappa & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad l = \begin{Vmatrix} 1+\kappa^2\chi^2 & \kappa^2\chi & 0 \\ \kappa^2\chi & \kappa^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

The principal values of the tensor l are specified by the equalities

$$2l_{1,2} = 1 + \kappa^2(1+\chi^2) \pm \sqrt{(1+\kappa^2(1+\chi^2))^2 - 4\kappa^2}, \quad l_3 = 1$$

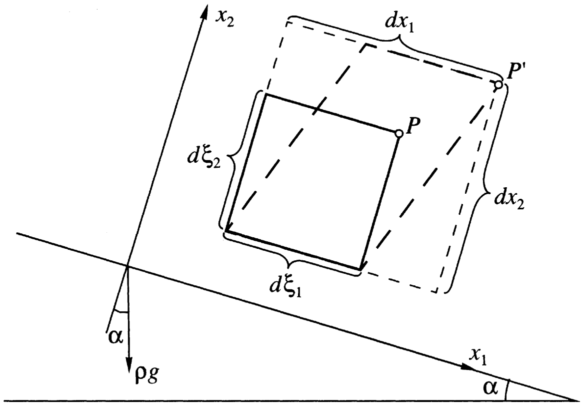

Fig. 5.

Since $l_1 l_2 = \kappa^2$, we have

$$\theta(h) = h_1 + h_2 = \ln\kappa, \quad \gamma(h) = \sqrt{\ln^2(l_1/\kappa) + \ln^2\kappa/3}$$

After some obvious reduction, the dilatancy equation can be written in the form

$$l_1 = \kappa^{1+\upsilon}, \quad \upsilon = \sqrt{\nu^2 + 1}$$

Solving it for χ, we obtain

$$\chi = \sqrt{(\kappa^{-1+\upsilon} - 1)(1 - \kappa^{-1-\upsilon})} \tag{5.2}$$

Taking into account equality (5.2), for the principal logarithmic strains we have

$$h_{1,2} = (1 \pm \upsilon)\ln\kappa/2, \quad h_3 = 0$$

For the tensors that appear on the right-hand side of formula (2.9), we obtain

$$l - l_{1,2}\delta = \left\| \begin{matrix} \kappa^{1\mp\upsilon} - \kappa^2 & \kappa^2\chi & 0 \\ \kappa^2\chi & \kappa^2 - \kappa^{1\pm\upsilon} & 0 \\ 0 & 0 & 1 - \kappa^{1\pm\upsilon} \end{matrix} \right\|$$

$$(l - l_1\delta)\cdot(l - l_2\delta) = \mathrm{diag}\{0, 0, (1 - \kappa^{1+\upsilon})(1 - \kappa^{1-\upsilon})\}$$

The separate differences table has the form

$$\begin{array}{c|lll}
\kappa^{1+\upsilon} & \dfrac{1+\upsilon}{2}\ln\kappa & & \\
 & & \dfrac{\upsilon\ln\kappa}{\kappa^{1+\upsilon} - \kappa^{1-\upsilon}} & \\
\kappa^{1-\upsilon} & \dfrac{1-\upsilon}{2}\ln\kappa & & \left(\dfrac{\upsilon}{\kappa^{1+\upsilon} - \kappa^{1-\upsilon}} - \dfrac{1}{2}\dfrac{1-\upsilon}{\kappa^{1-\upsilon} - 1}\right)\dfrac{\ln\kappa}{\kappa^{1+\upsilon} - 1} \\
 & & \dfrac{1-\upsilon}{2}\dfrac{\ln\kappa}{\kappa^{1-\upsilon} - 1} & \\
1 & 0 & &
\end{array}$$

By virtue of the equality (2.9), the non-zero components of the logarithmic tensor in the system of coordinates x_1, x_2 are as follows:

$$h_{11} = \left(\frac{1+\upsilon}{2} - \upsilon\frac{\kappa-\kappa^{-\upsilon}}{\kappa^{\upsilon}-\kappa^{-\upsilon}}\right)\ln\kappa, \quad h_{22} = \left(\frac{1+\upsilon}{2} - \upsilon\frac{\kappa^{\upsilon}-\kappa}{\kappa^{\upsilon}-\kappa^{-\upsilon}}\right)\ln\kappa$$

$$h_{12} = \frac{\upsilon\ln\kappa}{\kappa^{\upsilon}-\kappa^{-\upsilon}}\sqrt{(\kappa^{\upsilon}-\kappa)(\kappa-\kappa^{-\upsilon})}$$

The strain rates in the layer are calculated from the kinematic Eq. (5.1):

$$e_{22} = \dot{\kappa}/\kappa, \quad 2e_{12} = \dot{\kappa}\chi/\kappa + \dot{\chi}$$

The viscous stresses are specified by the equalities

$$\sigma_{11}^{\upsilon} = \sigma_{33}^{\upsilon} = (m - 2\eta/3)\dot{\kappa}/\kappa, \quad \sigma_{22}^{\upsilon} = (m + \eta/3)\dot{\kappa}/\kappa, \quad \sigma_{12}^{\upsilon} = \eta(\kappa\chi)^{\cdot}/\kappa \tag{5.3}$$

The quasi-static stresses are determined from the condition of proportionality between the deviators of the tensors σ^c and h and Eq. (2.8):

$$\frac{\sigma_{ii}^c}{p(\sigma^c)} = \left(\frac{h_{ii}}{\theta(h)} - \frac{1}{3}\right)\frac{2\varkappa^2}{f} - 1, \quad i = 1, 2, 3, \quad \frac{\sigma_{12}^c}{p(\sigma^c)} = \frac{2\varkappa^2 h_{12}}{\theta(h) f} \tag{5.4}$$

Neglecting the inertial forces in the case of slow motion of the material, from the equilibrium equations

$$\partial\sigma_{12}/\partial x_2 + \rho g\sin\alpha = 0, \quad \partial\sigma_{22}/\partial x_2 - \rho g\cos\alpha = 0$$

taking into account the formulae $dx_2 = \kappa d\xi_2$ and $\rho\kappa = \rho_0$, we obtain

$$\sigma_{12} = \sigma_{12}^{\upsilon} + \sigma_{12}^c = \rho_0 g(a_0 - \xi_2)\sin\alpha, \quad \sigma_{22} = \sigma_{22}^{\upsilon} + \sigma_{22}^c = -\rho_0 g(a_0 - \xi_2)\cos\alpha$$

(a_0 is the initial thickness of the layer). Using (5.3) and (5.4), we derive the ordinary differential equation

$$\frac{\eta(\kappa\chi)^{\cdot} - \kappa\sigma_{12}}{(m + \eta/3)\dot{\kappa} - \kappa\sigma_{22}} = \frac{\upsilon\sqrt{(\kappa^{\upsilon}-\kappa)(\kappa-\kappa^{-\upsilon})}}{((1+3\upsilon)/6 - f/(2\varkappa^2))(\kappa^{\upsilon}-\kappa^{-\upsilon}) - \upsilon(\kappa^{\upsilon}-\kappa)} \tag{5.5}$$

From equalities (5.2) it follows that

$$(\kappa\chi)^{\cdot} = \frac{(\upsilon+1)(\kappa^{\upsilon}-\kappa) + (\upsilon-1)(\kappa-\kappa^{-\upsilon}) + \upsilon'(\kappa^{\upsilon}-\kappa^{-\upsilon})\kappa\ln\kappa}{2\sqrt{(\kappa^{\upsilon}-\kappa)(\kappa-\kappa^{-\upsilon})}}\dot{\kappa}$$

where the prime denotes a derivative with respect to κ. Integrating Eq. (5.5) under the initial condition $\kappa(0) = 1$, we can obtain the function $\kappa(t, x_2)$.

In the initial stage of strain, when the parameter $\varkappa$ is limited from zero to a certain positive constant, formulae (5.4) for the stresses can be simplified, taking into account the asymptotic expansions

$$\kappa = 1 + \Delta, \quad \kappa^{\upsilon} - \kappa \approx (\upsilon - 1)\Delta, \quad \kappa - \kappa^{-\upsilon} \approx (\upsilon + 1)\Delta \quad (\Delta \ll 1) \tag{5.6}$$

We obtain

$$\frac{\sigma_{11}^c}{p(\sigma^c)} = \frac{\sigma_{33}^c}{p(\sigma^c)} = -1 - \frac{2\varkappa^2}{3f}, \quad \frac{\sigma_{22}^c}{p(\sigma^c)} = -1 + \frac{4\varkappa^2}{3f}, \quad \frac{\sigma_{12}^c}{p(\sigma^c)} = \frac{\varkappa^2\nu}{f} \tag{5.7}$$

Here, in accordance with the notation adopted above, $\nu = \sqrt{1/\varkappa^2 - 4/3}$. Eq. (5.5) also takes the simpler form

$$\frac{\eta\upsilon\dot{\kappa} - \rho_0 g(a_0 - \xi_2)\sin\alpha}{(m + \eta/3)\dot{\kappa} + \rho_0 g(a_0 - \xi_2)\cos\alpha} = \frac{\upsilon}{4/3 - f/\varkappa^2} \tag{5.8}$$

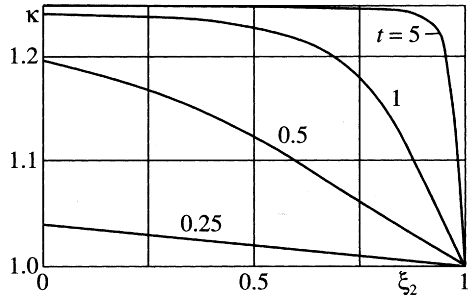

Fig. 6.

The material is in a state of limit equilibrium if $\dot{\kappa} = 0$. In this state, $\kappa = \kappa_0$ and $f = 1$; therefore,

$$\alpha = \alpha_0, \quad \operatorname{ctg}\alpha_0 = \upsilon_0, \quad \varkappa_0 = \sqrt{3}\sin\alpha_0/\sqrt{3+\sin^2\alpha_0}$$

The formula that relates the parameter $\varkappa_0$ to the angle of internal friction α_0 shows that for an adequate description of the surface of natural slope of a granular material within the model under consideration, the Mises–Schleicher cone must be inscribed in a Coulomb–Mohr cone.[10] This formula can also be obtained based on other approaches, i.e., using the limit equilibrium theory of granular materials.[16] In addition, it follows from it that the left-hand side of the first equality in (4.4) equals the coefficient of internal friction tan α_0.

The solution of Eq. (5.8) is correct only for $\alpha \geq \alpha_0$. For smaller angles, the equality $\varkappa\gamma(h) = \theta(h)$, whose right-hand side becomes negative because $\dot{\kappa} < 0$ and $\kappa < 1$, is meaningless. Motion under its own weight is impossible at such angles.

During the strain of a layer, when $\varkappa$tends to zero, the expansion formulae (5.6) become incorrect, and Eq. (5.8), therefore, does not hold. After changing to dimensionless quantities, apart from the slope and the characteristics of the dilatancy curve, the solution of Eq. (5.5) only depends on the ratio of the coefficients of viscosity m/η and on the dimensionless parameter $R = \rho_0 g a_0^2/(\eta v_0)$, which is equal to the ratio of the Reynolds number to the square of the Froude number (v_0 is the characteristic velocity of the particles in the direction of the x_2 axis). According to Eq. (5.5), $\dot{\kappa} \to 0$ near the free surface of the layer; therefore, dilatancy and shear flow occur predominantly in the lower part of the layer, in the region of increased pressure. The development of the process can be traced by analysing the variation of κ across the thickness of the layer.

Fig. 6 shows a series of curves that were obtained by the numerical fourth-order Runge–Kutta method for $R = 1$, $\alpha = 45°$, $m = 0$, $\varkappa_0 = 0.5$, $\kappa_* = 1.25$, $n = 0.5$, and various values of the dimensionless time t. The derivative v' was calculated in accordance with the power-law dependence (1.8):

$$\upsilon' = \frac{n}{\varkappa^2\upsilon(\kappa_* - \kappa)}$$

The calculations showed that the qualitative picture of the flow remains unchanged as R increases.

The tendency of the coefficients of viscosity to approach zero ($R \to \infty$) transforms Eq. (5.8) into the non-linear algebraic equation

$$\operatorname{tg}\alpha = \frac{\upsilon}{f/\varkappa^2 - 4/3}$$

which has no solutions for $\alpha > \alpha_0$, because the right-hand side of this equation does not exceed $1/\nu_0 = \tan\alpha_0$ by virtue of the inequalities $f > 1$ and $\kappa < \kappa_0$. Thus, the problem of the slow motion of a non-viscous medium is ill-posed without taking into account the inertial forces. In this case, the shear zone clearly degenerates into a line, and the layer slides over the inclined plane as a solid unit. The addition of inertial terms to the description of flow localization as R increases leads to a system of non-linear partial differential equations, whose solution is beyond the scope of this paper.

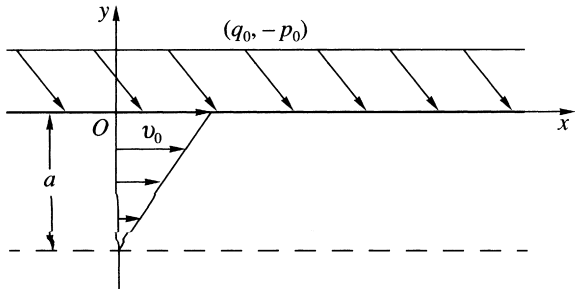

Fig. 7.

If the viscosity of the medium is non-zero, we have $\kappa \to \kappa_*$ and $\varkappa \to 0$ over the entire flow region on transferring to the steady regime. Eq. (5.5) is simplified ($a_* = a_0\kappa_*$ is the thickness of the dilatant layer):

$$\eta\dot{\chi} = \rho_* g(a_* - x_2)\sin\alpha$$

It describes motion with the quadratic velocity profile

$$v_1 = \rho_* g(2a_* x_2 - x_2^2)\sin\alpha/(2\eta) \tag{5.9}$$

It would be incorrect to take the limit as $\eta \to 0$ in (5.9), since this solution was obtained ignoring the inertial forces.

6. Plane-parallel motion

Let us consider the steady flow of a granular material occupying the half-space $y \le 0$, on whose boundary there is a heavy rough plate that moves in the direction of the horizontal x axis with a constant velocity v_0 (Fig. 7). Let p_0 be the pressure created by the weight of the plate, and let q_0 be the shear stress on the contact surface.

Flow occurs only if q_0 is greater than $q_0^* = p_0/\nu_0$, which corresponds to the state of limit equilibrium. When this state is reached, the medium dilates in a thin surface layer. The limit-equilibrium stresses are found from (5.7) when $\varkappa = \varkappa_0$ and $f = 1$. Taking into account the boundary condition $\sigma_y = -p_0$, we have

$$\sigma_x = \sigma_z = -p_0\left(1 + \frac{2}{\nu_0^2}\right), \quad \sigma_{xy} = \frac{p_0}{\nu_0} \tag{6.1}$$

Dilatancy is accompanied by the formation of a region of steady flow, i.e., a layer of finite depth a, on whose lower boundary the stress state (6.1) occurs on the dead-zone side, apart from to the replacement of p_0 by the pressure $p_1 = p_0 + \rho_* ga$. In the flow region, $\varkappa = 0$. Therefore, the stress tensor σ^c is spherical:

$$\sigma_x = \sigma_y = \sigma_z = -p_0 + \rho_* gy$$

The viscous stress tensor σ^v contains a single non-zero component $\sigma_{xy} = \eta\partial v_x/\partial y$, which is constant with depth and equal to q_0. Thus, the velocity profile in the flow region is linear in the thickness:

$$v_x = q_0(y + a)/\eta; \quad q_0 = \eta v_0/a$$

The shear stress is continuous on passing through the interface; we therefore have the quadratic equation

$$p_0 a + \rho_* g a^2 = \eta v_0 \nu_0$$

whose positive root is

$$a = \left(\sqrt{p_0^2 + 4\eta\rho_* g v_0 \nu_0} - p_0\right)/(2\rho_* g) \tag{6.2}$$

In the dead zone the stress state of the medium is found by invoking the equilibrium equations and the von Mises–Schleicher condition. Taking into account the equality $\sigma_x = \sigma_z$, which follows from the constitutive relations

(2.2) when $h_x = h_z = 0$, we obtain

$$\sigma_x = \sigma_z = -p_2\left(1+\frac{2}{\nu_0^2}\right)+\frac{\sqrt{3(p_2^2-p_1^2)}}{\varkappa_0\nu_0^2}, \quad \sigma_y = -p_2, \quad \sigma_{xy} = \frac{p_1}{\nu_0}$$

where $p_2 = p_1 - \rho_0 g(y+a)$.

For an experimental determination of the coefficients of internal friction and viscosity, it is convenient to use the equivalent form of the solution

$$\frac{1}{\nu_0} = \frac{q_0}{p_0+\rho_* g a}, \quad \eta = \frac{q_0 a}{\upsilon_0}$$

It turns out that if the plate is weightless ($p_0 \to 0$), the weight of the medium creates a viscous flow layer with thickness

$$a = \sqrt{\eta\upsilon_0\nu_0/(\rho_* g)}$$

When the viscosity is small, the asymptotic expression $a \approx \eta\upsilon_0\nu_0/p_0$, which shows that as η decreases, localization of the flow region occurs, and the magnitude of the shear stress q_0 tends to zero.

According to the solutions presented, the viscous properties have a significant effect on the motion of a granular material. In particular, localization and degeneration of the flow regions can occur as the coefficient of viscosity decreases.

In conclusion, we note that the proposed mathematical model has some advantages over the existing models. In the constitutive relations of a granular material, the relation between the intensity of the shear and the bulk strain is usually postulated in the form of the dilatancy equation $\gamma = F(\theta)$ (see, for example, Refs. 17,18). When such a relation is used, the isotropic volume expansion of the medium is described incorrectly, i.e., it is impossible in the absence of shear strain. Assignment of the relation using the inequality $\gamma \le F(\theta)$, which appears in the definition of the cone of permissible strains C, and the use of Hencky's logarithmic tensor enabled us to obtain a uniform description of the compacted and loosened states of a granular material without constraining the shear intensity. In addition, the model has a fairly simple mathematical structure, which is necessary for developing universal computational algorithms for solving boundary-value problems.

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